Quiz 2 Solution

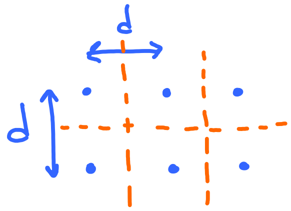
Tuesday, August 20, 2013
3:38 PM

Find $P(\mathcal{E})$ for each of the following constellation

Assume equiprobable message.

$N \sim \mathcal{N}(0, 2^2)$ $\qquad d = 4$

(a)



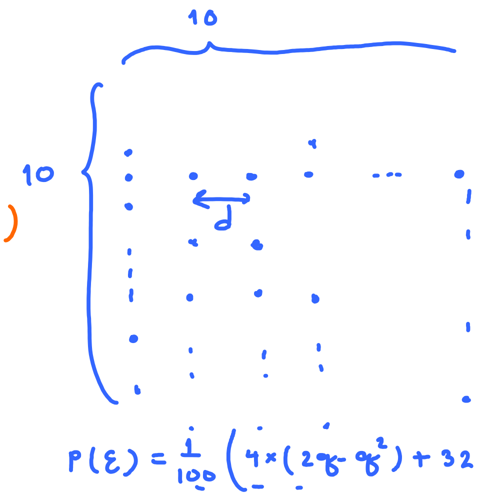
$(M = 6)$ $\qquad q = Q\left(\frac{d}{2\Delta}\right) = Q\left(\frac{4}{2\times 2}\right)$

$$P(\mathcal{E}) = \frac{1}{6}\left(4\times(2q - q^2) + 2\times(3q - 2q^2)\right)$$

$$= \frac{1}{6}\left(14q - 8q^2\right)$$

$$= \frac{7}{3}Q(1) - \frac{4}{3}Q^2(1)$$

(b)

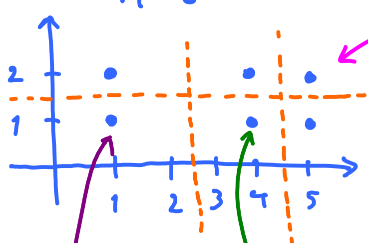


$(M = 100)$

$$P(\mathcal{E}) = \frac{1}{100}\left(4\times(2q - q^2) + 32(3q - 2q^2) + 64(4q - 4q^2)\right)$$

$$= \frac{1}{100}\left(360q - 324q^2\right) = 3.6\,Q(1) - 3.24\,Q^2(1)$$

(c) $\quad M = 6$



$$P(\mathcal{E}) = 1 - (1 - q_1)(1 - q_1)\ ; \quad q_1 = Q\left(\frac{1/2}{2}\right) = Q\left(\frac{1}{4}\right)$$

$$= 1 - (1 - 2q_1 + q_1^2)$$

$$= 2q_1 - q_1^2$$

$$P(\mathcal{E}) = 1 - (1 - q_1)(1 - q_2)$$

$$= 1 - (1 - q_1 - q_2 + q_1 q_2)$$

$$= q_1 + q_2 - q_1 q_2$$

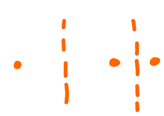
$$P(\mathcal{E}) = 1 - (1 - q_1 - q_2)(1 - q_1)\ ; \quad q_2 = Q\left(\frac{3/2}{2}\right) = Q\left(\frac{3}{4}\right)$$

$$= 1 - (1 - q_1 - q_2 - q_1 + q_1^2 + q_1 q_2)$$

$$= 2q_1 + q_2 - q_1^2 - q_1 q_2$$

$$P(\mathcal{E}) = \frac{1}{3}\begin{pmatrix} 2q_1 & - q_1^2 & & \\ q_1 & & + q_2 & - q_1 q_2 \\ 2q_1 & - q_1^2 & + q_2 & - q_1 q_2 \end{pmatrix} = \frac{1}{3}\left(5q_1 + 2q_2 - 2q_1^2 - 2q_1 q_2\right)$$

Alternatively 1D



$$P(\mathcal{E}) = \frac{1}{3}\left(q_2 + q_1 + q_2 + q_1\right)$$

$$= \frac{2}{3}(q_1 + q_2)$$

$$P(\mathcal{E}) = q_1$$

$$P(\mathcal{E}) = 1 - \left(1 - \frac{2}{3}(q_1 + q_2)\right)\left(1 - q_1\right)$$

$$= \cancel{1} - \left(\cancel{1} - \frac{2}{3}\gamma_1 - \frac{2}{3}\gamma_2 - \gamma_1 + \frac{2}{3}\gamma_1^2 + \frac{2}{3}\gamma_1\gamma_2\right)$$

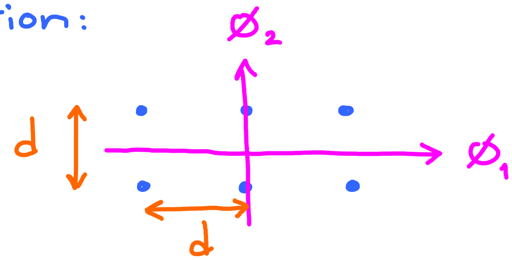
$$= \frac{5}{3}\gamma_1 + \frac{2}{3}\gamma_2 - \frac{2}{3}\gamma_1^2 - \frac{2}{3}\gamma_1\gamma_2$$

Quiz 3 Solution

Thursday, September 05, 2013   2:46 PM

Quiz 3 Solution

Express $\frac{d}{2\sigma}$ in the form of $\frac{E_b}{N_0}$ for the following constellation:



Assume equiprobable message.

$$E_s = \frac{1}{6}\left(2\times\left(\frac{d}{2}\right)^2 + 4\times\left(\left(\frac{d}{2}\right)^2 + d^2\right)\right) = \frac{1}{6}\left(\cancel{2}\frac{d^2}{\cancel{4}_2} + \overset{2}{\cancel{4}}\times\frac{5d^2}{\underset{2}{\cancel{4}}}\right) = \frac{11}{12}d^2$$

$$E_b = \frac{1}{\log_2 6}\frac{11}{12}d^2 \Rightarrow d^2 = \frac{12}{11}(\log_2 6)E_b$$

$$\frac{d}{2\sigma} = \sqrt{\frac{d^2}{4\sigma^2}} = \sqrt{\frac{d^2}{2N_0}} = \sqrt{\frac{6}{11}(\log_2 6)\frac{E_b}{N_0}}$$

Therefore, for the constellation above,

$$P(\varepsilon) = \frac{7}{3}q - \frac{4}{3}q^2 \quad \text{where} \quad q = Q\left(\sqrt{\frac{6}{11}(\log_2 6)\frac{E_b}{N_0}}\right)$$

Quiz 4 Solution

Tuesday, September 17, 2013 3:44 PM

① Suppose $P_{X,Y} = \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}$. Find $I(X;Y)$. ≈ 0.02

(4.5 pt)

$$H(X,Y) = H\left(\left[\tfrac{1}{5}\ \tfrac{2}{5}\ \tfrac{1}{5}\ \tfrac{1}{5}\right]\right) = 1.9219$$

$$P_{X,Y} = \begin{bmatrix} 1/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix} \begin{matrix} \to 3/5 \\ \to 2/5 \end{matrix} \Bigg\} \Rightarrow H(X) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) = 0.9710$$

$$\downarrow \quad \downarrow$$

$$\tfrac{2}{5} \quad \tfrac{3}{5} \quad \Rightarrow H(Y) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) = 0.9710$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y) = 0.019973 \approx 0.0200.$$

② Suppose $p_X(x) = \begin{cases} 1/5, & x=0 \\ 4/5, & x=1 \\ 0, & \text{otherwise} \end{cases}$ and

$P_{Y|X}$

x \ y	0	1
0	1/5	4/5
1	2/5	3/5

(4.5 pt)

Find $I(X;Y)$. ≈ 0.0215

$P_{Y|X}$

x \ y	0	1
0	1/5	4/5
1	2/5	3/5

$$\to H(Y|0) = H\left(\left[\tfrac{1}{5}\ \tfrac{4}{5}\right]\right) \approx 0.7219$$
$$\to H(Y|1) = H\left(\left[\tfrac{2}{5}\ \tfrac{3}{5}\right]\right) \approx 0.9710$$
$$\Rightarrow H(Y|X) = \tfrac{1}{5} \times H(Y|0) + \tfrac{4}{5} \times H(Y|1) \approx 0.9211$$

$$\overbrace{\begin{bmatrix} 1/5 & 4/5 \end{bmatrix}}^{P_X} \overbrace{\begin{bmatrix} 1/5 & 4/5 \\ 2/5 & 3/5 \end{bmatrix}}^{P_{Y|X}} = \overbrace{\begin{bmatrix} \tfrac{9}{25} & \tfrac{16}{25} \end{bmatrix}}^{P_Y} \Rightarrow H(Y) = H\left(\left[\tfrac{9}{25}\ \tfrac{16}{25}\right]\right) \approx 0.9427$$

$$I(X;Y) = H(Y) - H(Y|X) \approx 0.0215$$

Alternatively,

$P_{Y|X}$

x \ y	0	1
0	1/5	4/5
1	2/5	3/5

$\xrightarrow{\times \frac{1}{5}}$ (row 0), $\xrightarrow{\times \frac{4}{5}}$ (row 1)

x \ y	0	1
0	1/25	4/25
1	8/25	12/25

$$H(X,Y) = H\left(\left[\tfrac{1}{25}\ \tfrac{4}{25}\ \tfrac{8}{25}\ \tfrac{12}{25}\right]\right) \approx 1.6431$$

$$\begin{array}{c|cc} 0 & 1/5 & 4/5 \\ 1 & 2/5 & 3/5 \end{array} \xrightarrow[\times \frac{4}{5}]{} \begin{array}{c|cc} 0 & \frac{\ }{25} & \frac{\ }{25} \\ 1 & \frac{8}{25} & \frac{12}{25} \end{array}$$

$H(X,Y) = H([\ \ldots\ 25\ \ 25\ \ 25\ \ 25\ ])$

≈ 1.6431

$$\downarrow \qquad \downarrow$$

$$\frac{9}{25} \qquad \frac{16}{25}$$

$H(X) = H\left(\left[\frac{1}{5}\ \ \frac{4}{5}\right]\right) \approx 0.7219$

$\Rightarrow H(Y) = H\left(\left[\frac{9}{25}\ \ \frac{16}{25}\right]\right) \approx 0.9427$

$I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.0215$

*③ (1 pt) Suppose $p_{Y|X}$: $\begin{bmatrix} 1/5 & 4/5 \\ 1/5 & 4/5 \end{bmatrix}$ Find $I(X;Y)$.

Remark: Normally, to calculate $I(X;Y)$ you will need both p_X and $p_{Y|X}$.
Here, there must be something special about $p_{Y|X}$ that allows you to get $I(X;Y)$ without p_X.

Intuition: The rows of $p_{Y|X}$ are all the same. This implies that knowing the value of x does not change the pmf of Y.

Since X does not give any information about Y, we expect $I(X;Y) = 0$.

Direct calculation:

$H(Y|x) = H\left(\left[\frac{1}{5}\ \ \frac{4}{5}\right]\right) \approx 0.7219$ for any x.

So, $H(Y|X) = \sum_x p_X(x) H(Y|x) \approx 0.7219 \underbrace{\sum_x p_X(x)}_{1} \approx 0.7219$.

$I(X;Y) = H(Y) - H(Y|X)$ So, we need $H(Y)$ which in turn need p_Y.

Let's try $p_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$

Then,

$$\underset{p_X}{[1-p\ \ \ p]} \ \underset{}{\overset{p_{Y|X}}{\begin{bmatrix} 1/5 & 4/5 \\ 1/5 & 4/5 \end{bmatrix}}} = \overset{p_Y}{\left[\frac{1}{5}\ \ \frac{4}{5}\right]} \Rightarrow H(Y) = H\left(\left[\frac{1}{5}\ \ \frac{4}{5}\right]\right) = H(Y|X)$$

regardless of the value of p

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.

Quiz 5 Solution

Tuesday, September 24, 2013 3:50 PM

Evaluate the capacity values for the channels whose transition probabilities are given by each of the following matrices Q.

(a) $Q = \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$

(b) $Q = \begin{bmatrix} 1/6 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 1/2 & 1/6 \end{bmatrix}$

(c) $Q = \begin{bmatrix} 2/3 & 1/6 & 1/6 \\ 2/3 & 1/6 & 1/6 \end{bmatrix}$

(d) $Q = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(a) The channel is symmetric. So,

$$C = \log_2 |S_Y| - H(\underline{r}) = \log_2 2 - H\left(\left[\tfrac{1}{3}\ \tfrac{2}{3}\right]\right)$$

$$\approx 1 - 0.9183 \approx 0.0817.$$

(b) The channel is symmetric. So

$$C = \log_2 |S_Y| - H(\underline{r}) = \log_2 3 - H\left(\left[\tfrac{1}{6}\ \tfrac{1}{3}\ \tfrac{1}{2}\right]\right)$$

$$\approx 0.1258.$$

(c) Note that the rows of Q are all the same.

So, $H(Y|x) = H\left(\left[2/3\ \ 1/6\ \ 1/6\right]\right)$ for any x.

Therefore, $H(Y|X) = \sum_x p(x) H(Y|x) = H\left(\left[\tfrac{2}{3}\ \tfrac{1}{6}\ \tfrac{1}{6}\right]\right) \approx 1.2516$

Next, note that for any $p(x)$, we have

$$q(y) = \sum_x p(x) \underbrace{Q(y|x)} = \underbrace{Q(y|x)}.$$

For a given y, this is the same for all x.

So, $H(Y) = H(Y|x)$ as well.

$I(X;Y) = H(Y) - H(Y|X) = 0$ for any $\underline{p}$.

Remark. We can also conclude from $q(y) = Q(y|x)$ that $X \perp\!\!\!\perp Y$.
This also implies $I(X;Y) = 0$.

$$C = \max_{\underline{p}} I(X;Y) = 0.$$

(d) Note that this is a noisy channel with nonoverlapping outputs.

So, $C = \log_2 |S_X| = \log_2 3 \approx 1.5850$.